Limit probabilities of random Boolean expression values

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Boolean expressions

Expressions, values, trees.

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$$\underset{\text{expression}}{(1 \wedge \bar{0}) \vee (0 \vee 1)} \quad \underset{\text{value}}{= 1}$$

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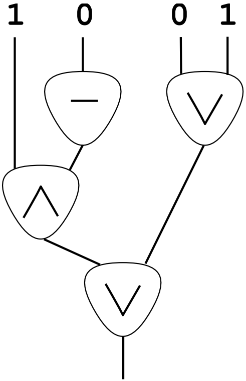
expression value



Complexity = number of nodes.

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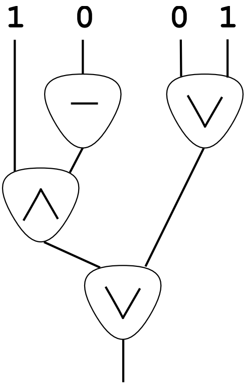

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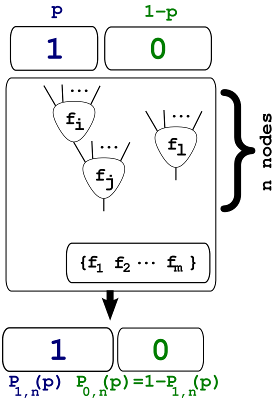
Randomization.



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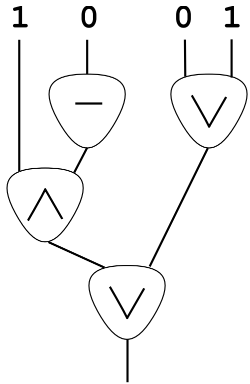

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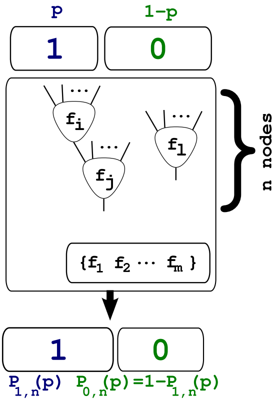

Probability function

$$P_1(p) = \lim_{n\to\infty} P_{1,n}(p).$$

Related research

Probability and complexity of functions computed by random Boolean formulas:

- Lefmann H., Savický P. Some typical properties of large AND/OR Boolean formulas, 1997.
- Chauvin B., Flajolet Ph., Gardy D., Gittenberger B. And/Or trees revisited, 2004.

Probability amplification by Boolean functions:

- Goldman S., Kearns M., Schapire R. Exact identification of read-once formulas using fixed points of amplification functions, 1993.

The method of investigation

- Truth tables for basis functions

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- Characteristic and basis polynomials

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- Formal languages and Schutzenberger's method

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- Probability functions

Characteristic and basis polynomials

- Examples

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Truth tables

x	y	$x \wedge y$	$x \vee y$	$x \operatorname{xor} y$
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

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Characteristic polynomials

$$A_\wedge(T,F) = T^2$$
$$A_\vee(T,F) =$$
$$= TF + FT + T^2 = 2TF + T^2$$
$$A_{\mathrm{xor}}(T,F) = TF + FT = 2TF$$

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- General case.

For $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$ let $|\tilde{\alpha}| = \#\{i : \alpha_i = 1\}$.

Then $A_{f(x_1,\ldots,x_n)}(T,F) = \sum\limits_{\tilde{\alpha}\in\{0,1\}^n} f(\tilde{\alpha}) T^{|\tilde{\alpha}|} F^{n-|\tilde{\alpha}|}$.

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- Characteristic polynomial for basis $B = \{f_1, \ldots, f_{|B|}\}$:
  $A(T, F) = \sum\limits_i A_{f_i}$.

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- Characteristic polynomial for basis $B = \{f_1, \ldots, f_{|B|}\}$:
  $A(T,F) = \sum\limits_i A_{f_i}$.
- Basis polynomial. For basis B let B_k denote the subset of functions with exactly k variables. The polynomial for B: $B(S) = \sum\limits_k |B_k| S^k$.

Probability functions

Theorem

Let B be a basis and $B(S)$, $A(T,F)$ its basis and chararcteristic polynomials. Then $\forall p \in (0,1)$ the limit $P_1(p) = \lim_{n\to\infty} P_{1,n}(p)$ exists and:

$$P_1(p) = \frac{A'_F(\tau, \sigma - \tau)}{\omega^{-1} - A'_T(\tau, \sigma - \tau) + A'_F(\tau, \sigma - \tau)},$$

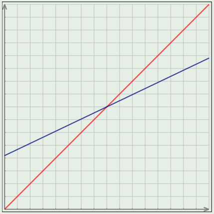
where ω and σ are the solution to

$$\begin{cases} \sigma = 1 + \omega B(\sigma) \\ 1 = \omega B'(\sigma) \end{cases}$$

with the least $|\omega|$, and $\tau = \tau(p)$, $0 \le \tau \le \sigma$ is the unique solution of $\tau = p + \omega A(\tau, \sigma - \tau)$.

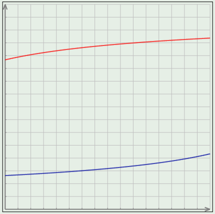
Probability functions

Example



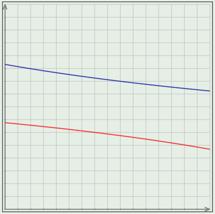
$B = \{\wedge, \vee\}$:
$P_1(p) = p$.

$B = \{\wedge, \vee, \bar{x}\}$:
$P_1(p) = \frac{7+2\sqrt{6}}{25}(p + 3 - \sqrt{6})$.



$B = \{\vee, \bar{x}\}$:
$P_1(p) = 1 - \frac{1}{2\sqrt{2+\sqrt{2}+(3+2\sqrt{2})p}}$.

$B = \{\wedge, \bar{x}\}$:
$P_1(p) = \frac{1}{2\sqrt{5+3\sqrt{2}-(3+2\sqrt{2})p}}$.



$B = \{\mathrm{nor}\}$:
$P_1(p) = 1 - \frac{1}{\sqrt{3-p}}$.

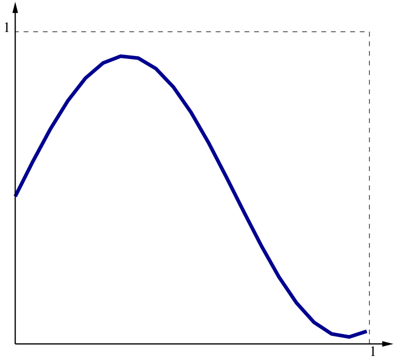
$B = \{\mathrm{nand}\}$:
$P_1(p) = \frac{1}{\sqrt{2+p}}$.

Approximation by probability functions

Theorem

Let $f(p)$ be a continuous function $f : [0,1] \to [0,1]$. For any $\varepsilon > 0$ there exists a basis B with probability function $P_1(p)$, such that for every $p \in [0,1]$:

$$|f(p) - P_1(p)| < \varepsilon.$$



Proof

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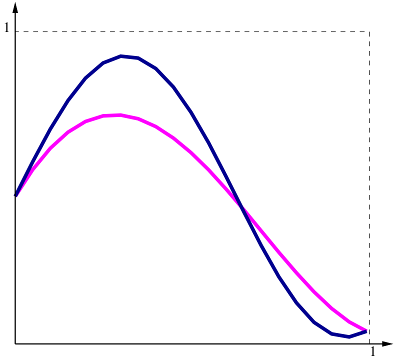

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- $|f(p)-\beta_r(f,p)|\le\varepsilon/4$

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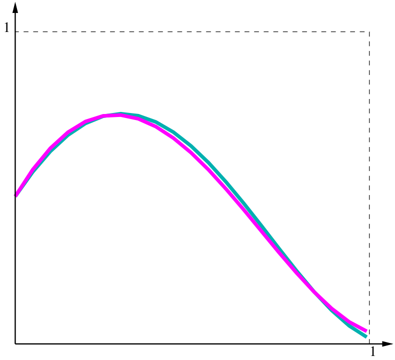

Proof

- $|f(p) - \beta_r(f,p)| \le \varepsilon/4$
- $|\beta_r(f,p) - \alpha(p)| \le \varepsilon/4$

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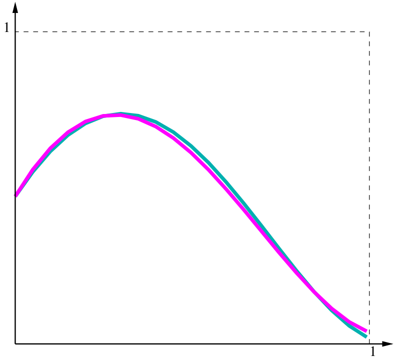

Proof

- $|f(p) - \beta_r(f,p)| \leq \varepsilon/4$
- $|\beta_r(f,p) - \alpha(p)| \leq \varepsilon/4$
- $\alpha(p) = \frac{A(p,1-p)}{|B|}$

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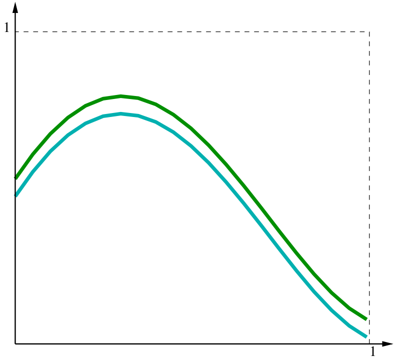

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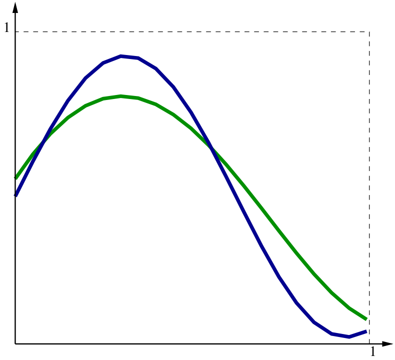

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- $|f(p) - P_1(p)| \le \varepsilon$

That's all.

Thank You for Your attention!